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THE ITERATION OF FUNCTIONS OF ONE VARIABLE.

By Albert A. Bennett.

A large part of the theory of the iteration of functions of several variables is illustrated by the relatively simple case of functions of one variable, and it is to the study of this case that the present paper is devoted. Several distinct classes of topics immediately suggest themselves in this connection. One of these classes comprises subjects of a purely formal nature, and is concerned only with expansions in the neighborhood of certain significant points. Another class of topics deals with the iteration of a real function within an interval. Still a third treats of the total behavior of the transforming function $B(x)$, when the given function $A_1(x)$ is analytic and defined for all values of x. Except for certain simple cases the function $B(x)$ cannot be one-valued. The present paper deals almost exclusively with the first two of the three classes of subjects just mentioned. In the formal problem, certain new points of view, and new results and formulæ are obtained, although a large proportion of this part is a systematization of results already known but largely unrelated. The second part of this paper deals with the iteration of a real function, a question which seems to have been neglected up to date.* The third topic mentioned above, has not here been touched upon, except in the case of certain simple rational functions. A few known theorems concerning the iteration of functions of a single variable, have been omitted, in view of the fact that they seem to have little significance in connection with the subject of the iteration of functions of several variables. Only those subjects have here been discussed which appear as natural introductory material to the more general case.

For references to those parts of the formal problem which have already been treated in the literature, one may consult the following:

Pincherle, on "Functional Equations and Operations," in the Encyklopädie d. Math. Wiss., II, A 11, and Encyclopédie d. Sci. Math., II, 26.

E. Schroeder, Math. Ann., 2 (1870), p. 317, and 3 (1871), p. 296.

J. Farkas, Journ. de Math. (3), 10 (1884), p. 102.

G. Koenigs, Bull. Sc. Math. (2), 7 (1883), p. 340, and (3) 1 (1884), suppl., p. 14.

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* Since this paper was written, there has been a report on one phase of the case of real iteration by Mr. J. F. Ritt. Cf. Am. Math. Soc. Bull., 21 (1915), p. 379.

E. Podetti, Giorn. Mat. (2), 4 (1897), p. 264.

C. Formenti, Reale Ist. Lomb. Rend. (2), 8 (1875), p. 275.

A. N. Korkine, Bull. Sc. Math. (2), 6 (1882), p. 228.

C. Bourlet, Toulouse Ann. (1), 12 (1898), mém. no. 3, p. 1–3.

A. Grévy, Thesis, Paris, 1894, and Ann. Ec. Norm. (3), 11 (1894), p. 287.

L. Leau, Thesis, Paris, 1897.

E. M. Lemeray, C. R., 125 (1897), p. 524, also S. M. F. Bull., 26 (1898), p. 10.

O. Spiess, Math. Ann., 62 (1906), p. 226.

M. Koppe, Die Iteration des Sinus . . ., Berlin, 1909.

Pincherle and Amaldi, Le Operazione Distributive, Chap. XIV. This may be consulted in connection with part of the use here made of matrices.

Note. The matrices and iteration discussed in this paper have no relation to the matrices and " Schapira's Iteration " treated in several recent papers by L. v. David, in the J. fur Math.

PART ONE. THE ITERATION OF POWER SERIES.

Definition and Examples of Iteration.

2. Let there be given an analytic function, $A_1(x)$. The function $A_1[A_1(x)]$, we shall define as $A_2(x)$, and in general for n, a positive integer, we shall define $A_n(x)$ as identical with $A_{n-1}[A_1(x)]$, or, what amounts to the same thing, as identical with $A_1[A_{n-1}(x)]$. We may similarly denote the inverse of $A_1(x)$ by $A_{-1}(x)$. For two given functions $A_1(x)$, and $F_1(x)$ we may define a $G_1(x)$, so that $G_1(x) \equiv F_{-1}\{A_1[F_1(x)]\}$, or, in other words, so that $G_1(x)$ is the transform of $A_1(x)$ through $F_{-1}(x)$. The $G_1(x)$ and $A_1(x)$ will then be so related that for any positive integer, or even negative integer value of n, $G_n(x) \equiv F_{-1}\{A_n[F_1(x)]\}$. In particular, we may replace the consideration of $y = A_1(x)$ by that of $\eta = G_1(\xi)$, where η is obtained from y, and ξ from x, by the same linear fractional transformation.

It is natural to inquire whether we might not suppose $A_1(x)$ and $G_1(x)$, as given, and determine a transforming function F, such that $G_1(x) \equiv F_{-1}\{A_1[F_1(x)]\}$, or, as it may be written, $A_1[F_1(x)] \equiv F_1[G_1(x)]$. In particular, we may examine the case in which $G_1(x)$ is simply $x + 1$. The function F associated with A by this particular choice of G, we shall call B. We seek therefore to determine a function B, such that

$$A_1[B(x)] \equiv B(x + 1). \tag{1}$$

We might equally well have started with $G_1(x)$ as cx, where c is a constant, $\neq 0$, $\neq 1$, and have sought a function $E(x)$, such that

$$A_1[E(x)] \equiv E(cx). \tag{2}$$

The two problems are not, of course, independent. We need merely replace x in (2) by c^x, to obtain

$$A_1[E(c^x)] \equiv E(c^{x+1}),$$

so that $E(c^x)$ is of the form of $B(x)$. If a function $B(x)$ be found, we may define $A_n(x)$, as identical with $B_1[n + B_{-1}(x)]$, for n not only an integer, but, indeed, for all complex values of n, and similarly for $E(x)$. The definition of $A_n(x)$, will depend to a considerable degree upon the particular function B or E, that is selected, except for n an integer. We shall define $A_n(x)$ as the nth *iterate* of $A_1(x)$, and the process of repeatedly substituting $A_1(x)$ in place of x, or, in general, of determining $A_n(x)$ from $A_1(x)$, as the process of *iteration.*

We shall now give a few simple examples of iteration, for which no extended discussion will be necessary. For the degenerate case of $A_1(x)$ equal to a constant, equation (1) defines $B(x)$ as also equal to a constant. For $A_1(x) \equiv x$, $B(x)$ may be any periodic function $P(x)$, with unity for a period. For $A_1(x) \equiv x + 1$, $B(x)$ is of the form $x + P(x)$, where $P(x)$ is defined as above. For $A_1(x) \equiv cx$, $0 \neq c \neq 1$, $B(x)$ is of the form $e^{[x+P(x)]\log c}$, $P(x)$ as before. This includes such expressions as ac^x, where a is an arbitrary nonzero constant, and more generally, it includes an extensive class of elliptic theta functions. For $A_1(x)$, an arbitrary linear fractional function, we obtain, either directly, or by reducing to a normal form, a class of associated functions, $B(x)$, containing an arbirtary periodic function $P(x)$, with unity for a period.

If $A_1(x)$ be not a linear fractional function, then either $A_1(x)$ itself, or else its inverse, $A_{-1}(x)$, must be multiple-valued, which would suggest that no one-valued function, $B(x)$, could exist for such a function $A_1(x)$. This is not, however, the case. For example, if we define $A_1(x)$ by the implicit equation:

$$\left.\begin{array}{lll} c_{00} & + c_{10}x & + c_{20}x^2 \\ + c_{01}A_1(x) & + c_{11}xA_1(x) & + c_{21}x^2A_1(x) \\ + c_{02}A_1^2(x) & + c_{12}xA_1^2(x) & + c_{22}x^2A_1^2(x) \end{array}\right\} = 0,$$

where $c_{ij} = c_{ji}$, $i, j, = 1, 2, 3$, and where the determinant of the coefficients is different from zero, we shall have a non-degenerate quadri-quadric relation between $A_1(x)$ and x. For this choice of $A_1(x)$, there exists among the solutions of (1), a $B(x)$, which is a single-valued elliptic function. The corresponding $A_n(x)$ satisfies an analogous quadri-quadric relation for every particular choice of n, whether real or complex. This is the problem which arises in connection with Poncelet Polygons, and the addition theorem of elliptic functions.*

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* Cf. Cayley, Elem. Treat. on Ell. Funct., p. 340, and Halphen, Traité des Fonc. Ell., vol. II, Chaps. 9 and 10, and an article by the author in *Annals of Math.* (2), 16 (1915), p. 97–118.

The question arises as to what is the most general algebraic expression $A_1(x)$, which is defined by equating to zero a polynomial in $A_n(x)$ and x, whose coefficients vary with n, but whose degree is, in general, the same, and for which there exists a $B(x)$, satisfying (1), and one-valued all over the complex plane. This problem involves the question as to what types of algebraic curves admit of one to one continuous transformations into themselves. As is well known, the rational and the elliptic curves are the only ones admitting such transformations. The problem just proposed reduces to the study of addition theorems of rational and of elliptic functions. We shall not go further into this matter, but shall proceed immediately to a more general problem.

Significance of the Formal Iteration of Power Series.

3. If we confine ourselves to the immediate neighborhood of a finite value of x, the equation (1) imposes no restrictions on $B(x)$, and is without significance. It is only when x is in the neighborhood of infinity, that we can expect to draw formal inferences with regard to $B(x)$, as a consequence of the behavior of $A_1(x)$. We shall seek to determine a solution $B(x)$, which at least admits asymptotic representation along the positive real axis, using the term "asymptotic representation," in its widest sense. In particular, we shall require that $\lim_{x=+\infty} B(x)$ exists. By replacing $A_1(x)$, if necessary, by a linear transform of it, we may secure that $\lim_{x=+\infty} B(x) = 0$, or from (1), that $\lim_{x=0} A_1(x) = 0$, when x approaches zero along a suitable path. There is another reason, also, which induces us to require that $\lim_{x=0} A_1(x) = 0$ for at least one method of approach, a reason, which is not explicitly related to the nature of the transforming function $B(x)$. The problem of computing the coefficients of a power series representation of $A_2(x)$ from the coefficients of a series representing $A_1(x)$ requires that at least one point (e) be known, either finite or infinite, for which $A(e) = e$, unless we are to have recourse to infinite summations which cannot be convergent except when the circle of convergence for the series $A_1(x)$, includes the point corresponding to the constant term of this series.

The derivation of $B(x)$ from $A_1(x)$ involves two steps; the first, that of the purely formal derivation of the coefficients, the second, that of proving the convergence or divergence of the series formally obtained for $B(x)$. The first of these steps is not, however, entirely devoid of significance, even when the resulting series is divergent. The series

$$A_1(x) \equiv a_{11}x + a_{12}x^2 + \cdots + a_{1n}x^n + \cdots$$

may be divergent, and yet it may well happen that there exists an analytic function $F(x)$, with an isolated essential signularity at the origin, for which

$$\lim_{x=0}\left[\frac{1}{n!}\frac{d^nF(x)}{dx^n}\right] = a_{1n},$$

along a suitably chosen path of approach. This is the case presented by the classical illustration of $y = e^{-1/x^2}$, where the path may be taken as either real semi-axis. Were we to choose infinity instead of the origin as the point approached, we should have the usual case of asymptotic convergence.[*] The set of coefficients $a_{11}, a_{12}, a_{13}, \cdots, a_{1n}, \cdots$, determines what may be called a *differential element of infinite order.* Two distinct functions may admit of the same asymptotic representation for a given method of approaching infinity, or to use more general ideas, they may have the same differential element at a given point, for a given path of approach. The two functions then represent curves which along a given path in the complex plane, have contact at the given point, of infinite order. So far as the formal determination of the series is concerned, we may suppose ourselves, throughout, as interested only in "asymptotic" forms, i. e., in differential elements for which questions of convergence are without significance.[*] It has been shown and it may be immediately verified that the analytic functions represented by

$$\sum_{n=1}^{\infty}\frac{a_n}{n!}(1 - e^{-\frac{1}{b_nx^2}})x^n,$$

where $1 < b_n > | a_n |$, b_n and a_n being real, have in general, an essential singularity at the origin, but have for real values the common differential element $(a_1, a_2, \cdots, a_n, \cdots)$, at the origin.

It would, of course, be possible to consider series $A_1(x)$, with no absolute term, but with terms containing other than merely integral powers of x. These, however, we shall not investigate. Even in the case of direct power series, a classification of types will be found necessary. We might enunciate an apparently arbitrary classification, justified *a posteriori* by the different conclusions obtained. We shall however find that the types which arise are essentially those which are suggested by regarding iteration as a case of collineation in an infinite number of dimensions. It is this point of view that we shall now present.

The Iteration of Power Series as Matrix Multiplication. Classification of Types.

4. For many purposes, it is more convenient to replace the nonhomogeneous relation $y = A_1(x)$ by the pair of homogeneous equations, obtained

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[*] Cf. W. B. Ford, S. M. F. Bull, 39 (1911), p. 347.

[†] Cf. Kasner, "Conformal Geometry," Fifth Int. Cong. of Math. Proc., vol. II, p. 81 ff. The term, "differential element of infinite order," has been discussed by Kasner. See also J. F. Ritt, Bull. Am. Math. Soc., 21 (1915), p. 379.

by replacing x by x_1/x_0, and y by y_1/y_0. We shall indeed write

$$\begin{cases} y_1 = x_0 A_1\left(\dfrac{x_1}{x_0}\right), \\ y_0 = x_0. \end{cases}$$

We define $A_1^{(m)}(x)$ as the polynomial of the mth degree in x, which is obtained by neglecting all but the first m terms of $A_1(x)$. Now the first m terms of $A_n(x)$, for n a positive or negative integer, depend only upon the first m terms of $A_1(x)$, that is, $A_n^{(m)}(x)$ is determined from $A_1(x)$ by using only $A_1^{(m)}(x)$.

Every binary form of the mth degree in y_0, y_1, will be itself a binary form of the mth degree in x_0, x_1, together with an infinite series, every term of which contains x_0 to some positive power in the denominator. We shall now define $\xi_i^{(m)}$ and $\eta_i^{(m)}$ as follows:

$$\xi_i^{(m)} = x_0^i \cdot x_1^{m-i} \quad i = 0, 1, 2, \cdots, m,$$

$$\eta_i^{(m)} = \text{integral part of series for } y_0^i \cdot y_1^{m-i} \equiv x_0^i[x_0 A_1(x_1/x_0)]^{m-i}.$$

Here $\xi_i^{(m)}$ is a product transform of the variables x_0, and x_1, and $\eta_i^{(m)}$ is related to the product transform of y_0 and y_1.* The expressions $\eta_i^{(m)}$, $i = 0, 1, 2, \cdots, m$, are linear in the $m + 1$ homogeneous variables, $\xi_i^{(m)}$, $i = 0, 1, 2, \cdots, m$, and in particular $\eta_0^{(m)} = \xi_0^{(n)}$. The matrix of this linear transformation from the $(m + 1)\xi^{(m)}$'s to the $(m + 1)$, $\eta^{(m)}$'s we may represent by $\boldsymbol{A}_1^{(m)}$.

For any other series $y = H(x)$ we may make an analogous discussion, and denote by $\boldsymbol{H}^{(m)}$ the analogous matrix. The first m terms of $z = H[A_1(x)]$, where $H(0) = 0$, depend upon $\boldsymbol{A}_1^{(m)}$ and $\boldsymbol{H}^{(m)}$ alone. Let us now define $\zeta_i^{(m)}$ as the integral part of the series for $z_0^i.z_1^{m-1} \equiv y_0^i[y_0 H(y_1/y_0)]^{m-i}$, where $z_1 = y_0 H(y_1/y_0)$, and $z_0 = y_0$. We may determine the first m terms of $z = H[A_1(x)]$ in three steps, by first taking the m leading terms of $y = A_1(x)$, secondly, taking the m leading terms of $z = H(y)$, and thirdly, substituting the first m terms of $A_1(x)$ for y in the first m terms of $z = H(y)$, and taking the m leading terms of the result. This corresponds to considering, first, the symbolic equation $\eta^{(m)} = \boldsymbol{A}_1^{(m)}\xi^{(m)}$, secondly the symbolic equation $\zeta^{(m)} = \boldsymbol{H}^{(m)}\eta^{(m)}$, and finally, substituting $\boldsymbol{A}_1^{(m)}\xi^{(m)}$ for $\eta^{(m)}$ in the equation for $\zeta^{(m)}$. Since $\boldsymbol{A}_1^{(m)}$ and $\boldsymbol{H}^{(m)}$ are the matrices of linear transformations, the result of the substitution yields merely a new matrix of the same type, which is the usual matrix product, $\boldsymbol{H}^{(m)}\boldsymbol{A}_1^{(m)}$. If in particular $\boldsymbol{H}^{(m)} = \boldsymbol{A}_{n-1}^{(m)}$, then we see that $\boldsymbol{A}_n^{(m)} = \boldsymbol{A}_{n-1}^{(m)}\boldsymbol{A}_1^{(m)}$, where the product is the usual matrix product. The significance of this relation lies of course in the fact that the initial first minor of $\boldsymbol{A}_n^{(m+1)}$ is simply $\boldsymbol{A}_n^{(m)}$ itself.

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* Cf. A. Hurwitz, Math. Ann., 45 (1894), p. 388, for the question of product transforms.

The relation $G_1(x) = F_{-1}\{A_1[F_1(x)]\}$ is equivalent to the equation $\boldsymbol{G}_1^{(m)} = \boldsymbol{F}_1^{(m),-1}\boldsymbol{A}_1^{(m)}\boldsymbol{F}_1^{(m)}$, where $\boldsymbol{F}_1^{(m),-1}$ is the inverse of $\boldsymbol{F}_1^{(m)}$ in the notation of matrices. This is the type of equation that arises in the theory of the classification of the linear transformations $\boldsymbol{A}_1^{(m)}$. Although the matrix $\boldsymbol{F}_1^{(m)}$ is not of the most general form possible, still it is not surprising that many of the theorems concerning the normal forms of linear transformations have here an application. For the sake of concreteness, we now give the form of $\boldsymbol{A}_1^{(5)}$

$$\boldsymbol{A}_1^{(5)} \equiv \left\{\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{11}^2 & 2a_{11}a_{12} & 2a_{11}a_{13} + a_{12}^2 & 2a_{11}a_{14} + 2a_{12}a_{13} \\ 0 & 0 & 0 & a_{11}^3 & 3a_{11}^2a_{12} & 3a_{11}^2a_{13} + 3a_{11}a_{12}^2 \\ 0 & 0 & 0 & 0 & a_{11}^4 & 4a_{11}^3a_{12} \\ 0 & 0 & 0 & 0 & 0 & a_{11}^5 \end{matrix}\right\}.$$

The case of $\boldsymbol{A}_1^{(5)}$ is typical of $\boldsymbol{A}_n^{(m)}$ to the extent that the first row contains only zero except for a one in the initial position, the elements below the main diagonal are all zero, while the elements in the main diagonal are the successive powers of a_{n1}. The roots of the characteristic equation of the matrix $\boldsymbol{A}_1^{(m)}$ are clearly a_{11}^i, $i = 0, 1, 2, \cdots, m$. We shall now examine the normal forms for the linear transformations whose matrices are $\boldsymbol{A}_1^{(m)}$.

The first case to consider is that in which all of the roots of the characteristic equation are distinct. This is the case in which a_{11} is not an integral root of unity. We may without exception in this case* find matrices $\boldsymbol{E}^{(m)}$ by the method of undetermined coefficients, for which

$$\boldsymbol{A}_1^{(m)} = \boldsymbol{E}^{(m),-1}\tilde{\boldsymbol{A}}_1^{(m)}\boldsymbol{E}^{(m)},$$

where $\tilde{\boldsymbol{A}}_1^{(m)}$ has the same main diagonal as $\boldsymbol{A}_1^{(m)}$, but has only zeros above as well as below this diagonal. The matrix $\boldsymbol{E}^{(m)}$ has a single arbitrary constant which is determined completely, if, for example, we put $e_1 = 1$. The only invariant of $\boldsymbol{A}_1^{(m)}$ as m takes on successively greater integer values, is in this case a_{11}.

We next come to those nonsingular cases in which all of the roots of the characteristic equation coincide. This occurs when $a_{11} = 1$. We may now have as many as two algebraic and one arithmetic invariant. It is impossible to have more, however, no matter how great m be taken. When

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* This first case has been treated by nearly all of the writers on the subject, Schroeder, Korkine, Farkas, Koenigs, etc., already referred to. This use of matrices $\boldsymbol{A}_n^{(m)}$ was made for the first time by H. v. Koch, Bil. t. Sv. Vet. Ak. Hand. 1, Math. 25 (1900), mém. no. 5, p. 1–24.

$a_{1i} = 0$, for every $i > 1$, we have the case of the identity. We shall have in all other cases, an arithmetic invariant k, such that $a_{1i} = 0$, $1 < i \leq k$, while $a_{1,k+1} \neq 0$. This is an invariant in the sense that no matter how $\boldsymbol{F}^{(m)}$ be chosen, the positive integer k for $\boldsymbol{G}_1^{(m)}$ where $\boldsymbol{G}_1^{(m)} = \boldsymbol{F}^{(m),-1}\boldsymbol{A}_1^{(m)}\boldsymbol{F}^{(m)}$, will be the same as for $\boldsymbol{A}_1^{(m)}$. By the use of undetermined coefficients we may find a matrix $\boldsymbol{E}^{(m)}$ such that

$$\boldsymbol{A}_1^{(m)} \equiv \boldsymbol{E}^{(m),-1}\tilde{\boldsymbol{A}}_1^{(m)}\boldsymbol{E}^{(m)},$$

where this time we mean by $\tilde{\boldsymbol{A}}_1^{(m)}$ the matrix obtained from a series of the form

$$A_1(x) \equiv x + \tilde{a}_{1,k+1}x^{k+1} + \tilde{a}_{1,2k+1}x^{2k+1} + \cdots + \tilde{a}_{1,lk+1}x^{lk+1} + \cdots,$$

and where $(1/x)\tilde{A}_1(x)$ is a power series in x^k. Both $\tilde{a}_{1,k+1}$ and $\tilde{a}_{1,2k+1}$, are invariants of $A_1(x)$ in the sense that any two series $\tilde{A}_1(x)$ equivalent to the same series $A_1(x)$, must have the same $\tilde{a}_{1,k+1}$ and $\tilde{a}_{1,2k+1}$.*

Of the nonsingular cases there remain to be considered those in which some but not all of the roots of the characteristic equation coincide. For these cases a_{11} must be an integral root of unity and we may write $a_{11}^h = 1$, where h is the smallest positive integer for which this is true. But if $A_1(x)$ be of this form then $A_h(x)$ will be of the type already mentioned, since it will commence with $x + \cdots$. Thus we have as invariants at least two integers h and k, and two algebraic invariants $\tilde{a}_{h,k+1}$ and $\tilde{a}_{h,2k+1}$. Here k must be a multiple of h but need not be h itself. Furthermore by the use of the method of undetermined coefficients, these are found to be the only invariants.

The only other cases which can arise are the singular cases, and in these, $a_{11} = 0$. If every succeeding a_{1i}, $i = 2, 3, \cdots$, vanishes, the expression $A_1(x)$ vanishes identically and is independent of x. Let us suppose then that $a_{1i} = 0$, $i = 1, 2, \cdots, k - 1$, while $a_{1k} \neq 0$. In this case k will be an arithmetic invariant, and this is the only invariant.† When once a normal matrix $\tilde{\boldsymbol{A}}_1^{(m)}$ has been selected, the remaining problem is to express, if possible, $\tilde{\boldsymbol{A}}_n^{(m)}$ as a matrix whose elements are explicit analytic functions of n.

Synopsis of Types.

6. On the basis of the above discussion we are prepared to enunciate‡ a classification of power series of the form

$$A_1(x) \equiv a_{11}x + a_{12}x^2 + \cdots + a_{1r}x^r + \cdots$$

from the point of view of iteration, into eleven types. We shall later examine the various cases in greater detail.

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* For this and the following case, see Leau, loc. cit. Cf. also Kasner, loc. cit.

† This is the case particularly investigated by Grèvy, loc. cit.

‡ This is given here for the first time.

Case I. a_{11} is not an integral root of unity and does not vanish.

Type I*a*, $|a_{11}| < 1, \neq 0$.

Type I*b*, $|a_{11}| > 1$.

Type I*c*,[*] $|a_{11}| = 1$, but $a_{11}{}^h \neq 1, h = 1, 2, \cdots$.

Case II. $a_{11} = 1$.

Type II*a*, $a_{1i} = 0, i = 2, 3, \cdots$.

Type II*b*, $a_{1i} = 0$, $i = 2, 3, \cdots, k$, but $a_{1,\,k+1} \neq 0$, and $2\tilde{a}_{1,\,2k+1} = (k+1)\tilde{a}_1{}^2{}_{,\,k+1}$, where $\tilde{a}_{1,\,k+1}$ and $\tilde{a}_{1,\,2k+1}$ are the invariants of the normal form already mentioned.

Type II*c*, $a_{1i} = 0$, $i = 2, 3, \cdots, k$, but $a_{1,\,k+1} \neq 0$, and $2\tilde{a}_{1,\,2k+1} \neq (k+1)\tilde{a}_1{}^2{}_{,\,k+1}$, where $\tilde{a}_{1,\,k+1}$ and $\tilde{a}_{1,\,2k+1}$ are defined as above.

Case III. $a_{11} \neq 1$, but $a_{11}{}^h = 1$, where h is the smallest positive integer for which this is true.

Type III*a*, $a_{hi} = 0, i = 2, 3, \cdots$.

Type III*b*, $a_{hi} = 0$, $i = 2, 3, \cdots, k$, but $a_{h,\,k+1} \neq 0$, and $2\tilde{a}_{h,\,2k+1} = (k+1)\tilde{a}_h{}^2{}_{,\,k+1}$, these being defined as above.

Type III*c*, $a_{hi} = 0$, $i = 2, 3, \cdots, k$, but $a_{h,\,k+1} \neq 0$, and $2\tilde{a}_{h,\,2k+1} \neq (k+1)\tilde{a}_h{}^2{}_{,\,k+1}$, defined as above.

Case IV. $a_{11} = 0$.

Type IV*a*, $a_{1i} = 0, i = 1, 2, \cdots$.

Type IV*b*, $a_{1i} = 0, i = 1, 2, \cdots, k-1$, but $a_{1k} \neq 0$.

Commutative Series.

6. If the approximate structure of a series for $A_n(x)$ be once found in a more or less indefinite fashion, it is sometimes possible to obtain the coefficients of $A_n(x)$, as explicit functions of n, without first reducing $A_1(x)$ to a normal form $\tilde{A}_1(x)$. This is done by using the following considerations. If a series $A_n(x)$ exists, it must satisfy the commutative condition

$$A_n[A_1(x)] \equiv A_1[A_n(x)].$$

Whenever the solutions $C(x)$, of the equation,

$$C[A_1(x)] \equiv A_1[C(x)] \tag{3}$$

form a one-parameter family, we may seek to identify this parameter with the n of $A_n(x)$. If, however, the C's form a discrete set, or contain more than one parameter, we cannot obtain $A_n(x)$ by this means. In terms of matrices, we have the equation,

$$\boldsymbol{C}^{(m)}\boldsymbol{A}_1{}^{(m)} = \boldsymbol{A}_1{}^{(m)}\boldsymbol{C}^{(m)}$$

which obviously greatly restricts $\boldsymbol{C}^{(m)}$. It is found that except in those cases in which a_{11} is an integral root of unity, there always exists a unique one-parameter family of series $C(x)$, commutative with $A_1(x)$, in the above sense, provided that in the cases for which $a_{11} = 1$, we require also that $c_1 = 1$. In the cases in which $a_{11} = 0$, the series C, although existing, are not, in general, in integral powers of x.

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[*] For questions of convergence this yields three quite distinct subcases, as we shall see later.

The identification of the parameter, with n, is, of course, not unique, but involves an arbitrary periodic function of period unity. A particularly simple choice is in each case almost immediately obvious. For instance, to illustrate the most common case, let $0 \neq | a_{11} | \neq 1$, and let

$$C(x) = c_1x + c_2x^2 + c_3x^3 + \cdots$$

while $A_1(x)$ is denoted as heretofore. Then $C[A_1(x)]$ will be of the form,

$$\begin{array}{ccccccc} c_1a_{11}x & + & c_1a_{12}x^2 & + & c_1a_{13}x^3 & + & \cdots \\ & + & c_2a_{11}^2x^2 & + & 2c_2a_{11}a_{12}x^3 & + & \cdots \\ & & & + & c_3a_{11}^3x^3 & + & \cdots \\ & & & & & + & \cdots \end{array}$$

and $A_1[C(x)]$ will be obtained from this by interchanging a_{1i} and c_i, $i = 1, 2, 3, \cdots$. Equating coefficients of like powers of x, we obtain,

$$\begin{aligned} c_1a_{11} &= a_{11}c_1, \\ c_1a_{12} + c_2a_{11}^2 &= a_{11}c_2 + a_{12}c_1^2, \\ c_1a_{13} + 2c_2a_{11}a_{12} + c_3a_{11}^3 &= a_{11}c_3 + 2a_{12}c_1c_3 + a_{13}c_1^3, \\ \cdots\cdots & \cdots\cdots \end{aligned}$$

Hence

$$\begin{aligned} c_2 &= \frac{c_1}{a_{11}}\frac{(c_1 - 1)}{(a_{11} - 1)}a_{12}, \\ c_3 &= \frac{c_1}{a_{11}}\frac{(c_1^2 - 1)}{(a_{11}^2 - 1)}a_{13} + 2\frac{c_1}{a_{11}}\frac{(c_1 - 1)}{(a_{11} - 1)}(c_1 - a_{11})a_{12}^2. \\ \cdots & \cdots\cdots \end{aligned}$$

If $C(x)$ is to be identified with $A_n(x)$, we must have for integer values of n, $c_1 = a_{11}^n$. The most general form of c_1 is then, $a_{11}^{n+P(n)}$, where $P(n)$ is a periodic function of period unity, vanishing for n an integer. The most obvious choice of c_1 is undoubtedly that in which $P(n)$ vanishes identically, so that for all values of n, $c_1 = a_{11}^n$. Every succeeding c_i, $i = 2, 3, \cdots$, is then completely determined.

Discussion of the Different Types.

7. For any type under Case I, we may always determine a series $E(x)$, and if we put $e_1 = 1$, it will be determined uniquely and in such a way,

that

$$A_1(x) \equiv E_{-1}[a_{11}E(x)].$$

By writing $E_{-1}(a_{11}{}^x) \equiv B(x)$, we obtain a $B(x)$, for which

$$A_n(x) \equiv B[n + B_{-1}(x)],$$

as was desired. This gives $A_n(x)$ as an explicit function of n. We may obtain, however, another form of $A_n(x)$, which proves for many purposes much more convenient than that just given. It is to be expected that by the use of the expressions $A_1(x)$, $A_2(x)$, $\cdots$, $A_p(x)$, $\cdots$, where p is a positive integer, we may obtain a sort of interpolation formula, giving $A_n(x)$ for all values of n. We shall seek indeed a series of the following form

$$A_n(x) \equiv \sum_{i=0}^{\infty} \alpha_i U_i(x),$$

where $U_i(x)$ is independent of n, and is a power series in x commencing with a term in x^{i+1}, expressible as a linear function of $A_0(x)$, $A_1(x)$, $\cdots$, $A_i(x)$, with coefficients which are themselves polynomials in a_{11} with numerical coefficients, and where α_i is a function of a_{11} and n alone. These conditions serve to determine the U's and α's as follows:

$$\begin{aligned}
U_0(x) &= A_0(x) \equiv x,\\
U_1(x) &= A_1(x) - a_{11}A_0(x),\\
U_2(x) &= [A_2(x) - a_{11}A_1(x)] - a_{11}{}^2[A_1(x) - a_{11}A_0(x)],\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
U_i(x) &\equiv U_{i-1}[A_1(x)] - a_{11}{}^i U_{i-1}(x).\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

The coefficients, α_i, satisfy the recursion formula,

$$\alpha_i = \frac{1}{a_{11}} \frac{(a_{11}{}^{n-i+1} - 1)\alpha_{i-1}}{(a_{11}{}^i - 1)}, \quad \alpha_0 = a_{11}{}^n.$$

Then we will have

$$(4) \qquad \begin{aligned} A_n(x) \equiv a_{11}{}^n \Big[U_0(x) &+ \frac{1}{a_{11}} \frac{(a_{11}{}^n - 1)}{(a_{11} - 1)} U_1(x) \\ &+ \frac{1}{a_{11}{}^2} \frac{(a_{11}{}^n - 1)}{(a_{11} - 1)} \frac{(a_{11}{}^{n-1} - 1)}{(a_{11}{}^2 - 1)} U_2(x) + \cdots \Big]. \end{aligned}$$

To prove that the $A_n(x)$ determined by (4) is actually identical with that determined by the use of the series $B(x)$, we need only note the following facts:

1. Since $A_i(x)$ commences with the term $a_{11}{}^i x$, the series $U_i(x)$ must

necessarily contain no power of x less than the $(i+1)$st, as is readily verified by the use of the recursion formula for the U's.

2. The equation (4) is an identity for n, a positive integer, say $i-1$. For then,

$$\alpha_i = \frac{1}{a_{11}} \frac{(a_{11}{}^0 - 1)\alpha_{i-1}}{(a_{11}{}^i - 1)},$$

will vanish, and all succeeding α's, while the non-vanishing terms add up to exactly $A_i(x)$, as is readily verified.

3. As a consequence of the two facts just mentioned, the coefficient of x^p, in $A_n(x)$, p a positive integer, and n arbitrary, will depend upon only the first p, U's. It will therefore be a polynomial in $a_{11}{}^n$ and contain n in no other way, the coefficients being rational in a_{11} and integral, rational in the first p coefficients of $A_1(x)$.

4. The coefficients of x^p in $A_n(x)$, as obtained by the use of $B(x)$, will also satisfy the conditions just stated, and since these two polynomials in $a_{11}{}^n$ must coincide for an infinite number of values of $a_{11}{}^n$, viz. for $n = p+1$, $p+2, \cdots$, the polynomials are identical. Thus the two determinations of $A_n(x)$ coincide.

Another* and somewhat analogous representation for $A_n(x)$ may be obtained from (4), in which the variable n occurs in a more convenient form, but in which the coefficient of x^p, $p > 1$, is in general an infinite series in a_{11}. This is defined in terms of V's, as follows:

$$\begin{aligned} V_0(x) &\equiv A_0(x) \equiv x, \\ V_1(x) &\equiv A_1(x) - a_{11}A_0(x), \\ V_2(x) &\equiv [A_2(x) - a_{11}A_1(x)] - a_{11}[A_1(x) - a_{11}A_0(x)] \\ &\equiv A_2(x) - 2a_{11}A_1(x) + a_{11}{}^2A_0(x), \end{aligned}$$

. . . .

$$(5)\quad \begin{aligned} V_i(x) &\equiv V_{i-1}[A_1(x)] - a_{11}V_{i-1}(x) \\ &\equiv A_i(x) - \binom{i}{1}a_{11}A_{i-1}(x) + \binom{i}{2}a_{11}{}^2A_{i-2}(x) - \cdots \pm a_{11}{}^iA_0(x), \end{aligned}$$

. . . .

$$\begin{aligned} A_n(x) \equiv a_{11}{}^n\Big[V_0(x) &+ \binom{n}{1}\frac{1}{a_{11}}V_1(x) + \binom{n}{2}\frac{1}{a_{11}{}^2}V_2(x) + \cdots \\ &+ \binom{n}{i}\frac{1}{a_{11}{}^i}V_i(x) + \cdots\Big]. \end{aligned}$$

In Type IIa, $A_1(x)$ is the identity. This case requires no further mention. Type IIb is a well-known form discussed by many of the authors already mentioned. By the use, for example, of the method of undetermined coefficients, the $A_1(x)$ may here be transformed through a suitable $F(x)$ into a series of the form x times a power series in x^k, the leading term being exactly x. This applies also to Type IIc. There is a wide degree of freedom in the choice of the series. In fact only the coefficients of the terms in x, x^{k+1}, and x^{2k+1} are preassigned. For Type IIb, we may in particular take the series to be in the form

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* Equations (4) and (5) are given here for the first time.

$$y = \tilde{A}_1(x) = \frac{x}{\sqrt[k]{1 - k\tilde{a}_{1,\,k+1}x^k}}.$$

If in this expression we replace y by $y^{1/k}$, and x by $x^{1/k}$, we obtain

$$y = \frac{x}{1 - k\tilde{a}_{1,\,k+1}x}.$$

Replacing next, y by $1/y$, and x by $1/x$, we have

$$y = x - k\tilde{a}_{1,\,k+1},$$

so that in this way we have secured a transform of $y = A_1(x)$, in the form

$$y = c + x.$$

Putting $- k\tilde{a}_{1,\,k+1}y$ for y and $- k\tilde{a}_{1,\,k+1}x$ for x, we arrive at exactly

$$y = 1 + x,$$

as desired. We need merely retrace our steps to secure the transforming series $B(x)$.

In the case of Type IIc, the best that we can do for the first two steps is to transform $y = A_1(x)$ into

$$y = \frac{x}{1 - k\tilde{a}_{1,\,k+1}x - k\left[\tilde{a}_{1,\,2k+1} - \frac{1+k}{2}\tilde{a}_1{}^2{}_{,\,k+1}\right]x^2}.$$

This is an analytic function in the neighborhood of the origin, that resembles somewhat the analytic function

$$y = \frac{x}{1 - cx},$$

where c is a suitably chosen constant. Any transforming function, however, which carries the given function into one of the latter type, must have an essential singularity at the origin, it being understood that for Type IIc, $2\tilde{a}_{1,\,2k+1} \neq (k + 1)\tilde{a}_1{}^2{}_{,\,k+1}$. For a fairly complete discussion of the existence of this transforming function, *cf.* Leau, *loc. cit.* It will be unnecessary for us to go into this particular question, since a direct representation in series may be found without the use of a transforming function. We need only regard Case II, as a limiting form of Case I, and pass to the limit in either (4) or (5). We obtain in both cases the same series.* We shall have:

$$
\begin{aligned}
U_0(x) &\equiv A_0(x) \equiv x, \\
U_1(x) &\equiv A_1(x) - A_0(x), \\
U_2(x) &\equiv A_2(x) - 2A_1(x) + A_0(x), \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
U_i(x) &\equiv U_{i-1}[A_1(x)] - U_{i-1}(x), \\
&\equiv A_i(x) - \binom{i}{1} A_{i-1}(x) + \binom{i}{2} A_{i-2}(x) - \cdots \pm A_0(x), \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
$$

while for α_i we have

$$\alpha_i = \frac{n - i + 1}{i} \alpha_{i-1} = \binom{n}{2}, \qquad \alpha_0 = 1.$$

Hence, when $a_{11} = 1$, *i. e.*, for Case II,

$$(6) \quad A_n(x) \equiv A_0 + \frac{n}{1}(A_1 - A_0) + \frac{n(n-1)}{1 \cdot 2}(A_2 - 2A_1 + A_0) + \cdots.$$

For Case III, if $A_1(x)$ be given, there exist series $C(x)$, for which

$$C[A_1(x)] \equiv A_1[C(x)],$$

but these C's, instead of containing but a single parameter, contain more than one, and in some cases, contain an infinite number of parameters. For example $A_1(x) \equiv -x$ is commutative with any odd function of x. But if $C_1(x)$ be commutative with $A_1(x)$ then $C_m(x)$ is commutative with $A_n(x)$, where m and n are any positive integers, although the converse is not necessarily true. For c_{11} an integral root of unity, we may choose m and n so that $a_{m1} = 1$, and $c_{m1} = 1$. This carries us back to Case II. In particular, by taking $m = 1$, we may conclude that if $A_1(x)$ be such that a $C_1(x)$ exists for which c_{11} is equal to one, then $C_1(x)$ can contain, while c_{11} is kept equal to unity, at most one free parameter, unless $A_1(x)$ is of Type III*a*. Those of Type III*a* all contain an infinite number of parameters as we shall presently show. The same remarks apply when c_{11} is any integral power of a_{11}, say the l'th, since $C_1[A_{h-l}(x)]$ will also be commutative with $A_1(x)$, where h is the lowest positive integer for which $a_{11}^h = 1$, but $C_1[A_{h-l}(x)]$ will have unity for the coefficient of the first term. Apart from c_{11}, the only arbitrary parameter is a certain $c_{1,k+1}$, as is readily verified by the method of undetermined coefficients. Now x itself, is certainly commutative with $A_1(x)$, so that there exists at least one series $C_1(x)$ for which $c_{11} = 1$, $c_{1i} = 0$, $i = 2, 3, \cdots$. Since all series $C_1(x)$ with the same c_{11} coincide as far as the term containing the parameter $c_{1,k+1}$, it is clear that $A_h(x)$ which is commutative with $A_1(x)$, and for which $a_{h1} = 1$, is of the form

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* This series has been obtained directly by Schroeder, Math. Ann., 2 (1870), p. 317.

$$A_h(x) \equiv x + 0 + \cdots + 0 + a_{h,k+1}x^{k+1} + a_{h,k+2}x^{k+2} + \cdots.$$

The method of undetermined coefficients which gives the k to be used, shows us that this k is h itself, or an integral power of h. Hence had we started with a series

$$A_1(x) = x + 0 + \cdots + 0 + a_{1,k+1}x^{k+1} + a_{1,k+2}x^{k+2} + \cdots$$

where $a_{1,k+1} \neq 0$, then the only expressions $A_{(1/h)}(x)$, which can exist, will be for h, a factor of k. For types III*b* and III*c*, integer iterates, of course, always exist, and they will either be of the same type as the given series or of the analogous type under Case II. In some instances a few fractional iterates of the type of the given series may exist. However, no series $A_n(x)$ of Types III*a*, or III*b*, can exist, containing an arbitrary parameter, n.

Type III*a* demands further attention.* For the sake of definiteness, we shall first consider the case in which $h = 2$. Here $A_1(x)$ is such that if $y = A_1(x)$ then $x = A_1(y)$. Viewed geometrically, this gives us a curve in the x, y plane, whenever the series $A_1(x)$ is convergent. This curve is unaltered by interchanging x and y, that is, it is symmetrical about the line $y = x$. If we transform the coordinates by writing

$$u_0 = \tfrac{1}{2}(x - y),$$
$$u_1 = \tfrac{1}{2}(x + y),$$

the curve will be symmetrical about the line $u_0 = 0$, i. e., the u_1-axis, and in the neighborhood of the origin it will have contact with the u_0-axis, since the slope of the curve $y = A_1(x)$ is unity, at the origin. Thus, if in the neighborhood of the origin, u_1 be represented as a power series in u_0, for points on the curve, the series will represent an even function of u_0. Furthermore, this is the *only* restriction imposed upon u_1, as a function of u_0. We may select for u_1, an arbitrary even function of u_0, and determine x and y by the equations

$$x = u_0 + u_1,$$
$$y = -u_0 + u_1.$$

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* This leads to $A_h(x) = x$, which is known as "Babbage's Problem." Cf. the following: J. D. Gergonne, Ann. Math., 12 (1821–2), p. 73. O. Rausenberger, Math. Ann., 18 (1881), p. 379. E. Iaggi, Nouv. Ann. (3), 19 (1900), p. 483. M. J. van Ufen, Amst. Ak. Versl., 18 (1909), p. 860, and 19 (1910), p. 27–31. The discussion here is new.

If from these equations we solve for y as a function of x by eliminating the u_0 and u_1, the solution $y = A_1(x)$, will be such that also $x = A_1(y)$. We may forego the geometric picture, and select u_1 as an arbitrary series in even powers of u_0 without constant term, and not necessarily convergent. The series $A_1(x)$ obtained by solving as before, for y, will be determinate, and if $y = A_1(x)$, $x = A_1(y)$.

The discussion for a general integer value of h, follows along similar lines, although no simple geometric illustration of an obvious sort suggests itself. We shall first suppose a series $A_1(x)$ given for which $A_h(x) \equiv x$. From this series we shall determine h series u_i, $i = 0, 1, \cdots, h - 1$, which we shall find to be restricted in a very simple manner only. This done, we shall start with the h series u_i and so determine the form of the most general series $A_1(x)$ for which $A_h \equiv x$. Let us write

$$u_0 = \frac{1}{h}(A_0 + \omega A_1 + \omega^2 A_2 + \cdots + \omega^{h-1} A_{h-1}),$$

$$u_1 = \frac{1}{h}(A_0 + \omega^2 A_1 + \omega^4 A_2 + \cdots + \omega^{2h-2} A_{h-1}),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$u_{h-1} = \frac{1}{h}(A_0 + A_1 + A_2 + \cdots + A_{h-1}),$$

where ω is a primitive hth root of unity. We wish now to discover what restrictions must obtain among the u's in order that $A_h \equiv A_0$. If we replace A_i by A_{i+1}, u_0 is carried into $\omega' u_0$, where ω' is the reciprocal of ω, while u_{h-1} is unaltered. Hence if we express u_{h-1} as a power series in u_0, by eliminating the independent variable x, occurring in each A_i, $i = 0, 1, \cdots, h - 1$, we shall find that only the terms in powers of $u_0{}^h$ appear, for if any other term were to appear, it would be changed into the product of its original value by a power of ω, different from unity, which is impossible, since u_{h-1} is to remain unaltered. In a similar manner the power series for u_i, $i = 1, 2, \cdots, h - 1$, in terms of u_0, contain only terms in which $u_0{}^{i+1}$ times a power of $u_0{}^h$ appears. These are the only restrictions upon the series u_i in terms of u_0. We may suppose, if we desire, that the u's are given, and that we are to find $A_1(x)$. We may take any $h - 1$ power series, $P_i(t)$, $i = 1, 2, \cdots, h - 1$, convergent or divergent and write $u_i = u_0{}^{i+1} P_i(u_0{}^h)$. If we solve for the A's formally in terms of the u's, we obtain in particular

$$A_0(x) = x = (u_0 + u_1 + \cdots + u_{h-1})$$

i. e., $x = E_{-1}(u_0)$ where $E_{-1}(u_0)$ is, by definition, the series in u_0 obtained by replacing each u_i, $i = 0, 1, 2, \cdots, h - 1$ by its series representation in terms of u_0, in the right hand member of the above equation. By definition of E_{-1}, it follows that

$$u_0 = E(x).$$

Now $E_{-1}(u_0)$ is of the form

$$E_{-1}(u_0) \equiv u_0 + u_0^2P_1(u_0^h) + u_0^3P_2(u_0^h) + \cdots + u_0^hP_{h-1}(u_0^h).$$

Thus $E_{-1}(\omega u_0)$ is simply $(\omega u_0 + \omega^2u_1 + \omega^3u_2 + \cdots + u_{h-1})$, since when we replace u_0 in $u_i = u_0^{i+1}P_i(u_0^h)$, by wu_0, we obtain merely $\omega^{i+1}u_i$. But this expression for $E_{-1}(\omega u_0)$ is exactly what we obtain formally, when we solve for A_1 in terms of the u's. Thus we have $A_1 \equiv E_{-1}(\omega u_0)$ or

$$A_1(x) \equiv E_{-1}[\omega E(x)],$$

from which we verify that indeed $A_h(x) \equiv x$, since obviously for i any positive integer we have

$$A_i(x) \equiv E_{-1}[\omega^iE(x)].$$

Thus we verify that it is not only necessary but sufficient for $u_i(u_0)$ to be of the form $u_0^{i+1}P_i(u_0^h)$, in order that the $A_1(x)$ associated with the u's as above shall be such that $A_h(x) \equiv A_0(x)$. The $E_{-1}(u_0)$ that we have obtained above is a power series in u_0 characterized by the property that the absolute term is missing, the coefficient of u_0 is unity, and the coefficient of u_0^{lk+1}, $l = 1, 2, \cdots$, vanishes. For any given series $A_1(x)$ the series E_{-1} satisfying the conditions just mentioned is uniquely determined by the equation

$$A_1(x) \equiv E_{-1}[\omega E(x)]$$

analogous to the equations obtained in Case I. Unlike the situation in Case I, we have here imposed additional conditions on E_{-1}. If we replace u_0 by a power series $u_0 = S_{-1}(v)$, where S_{-1} is restricted to the extent of having the coefficient of every term which does not contain v to a power of the form $lk + 1$, $l = 0, 1, 2, \cdots$, vanish, while the remaining coefficients are arbitrary, then we obtain

$$A_i(x) \equiv E_{-1}\{S_{-1}[\omega^iS\{E(x)\}]\},$$

where A_i, E, and ω are fixed and S_{-1} has an infinite number of arbitrary coefficients as already remarked. We may think of $E_{-1}[S_{-1}(v)]$ as constituting a new series $F_{-1}(v)$, for which the coefficients of v^{lk+1}, $l = 1, 2, \cdots$ are arbitrary, the others being then determined, if A_1 be given, so that

$$A_1(x) = F_{-1}[\omega F(x)].$$

Case IV presents little of interest. Type IV*a* is trivial, and will be passed over without further comment. Type IV*b* has a general iterate as a function of n, but it is not expressible as a power series in integral powers of x; terms with irrational exponents, for example, must be introduced. We may however obtain a transforming function, which reduces this case to Case I. We first choose α, as one of the $(k-1)$st roots of a_{1k}, and replace y by y/α, and x by x/α. We obtain in this way a new series of Type IV*b*, but one in which a_{1k} is now unity. By the method of undetermined coefficients, we may find a series $F(x)$ such that

$$F[A_1(x)] \equiv [F(x)]^k,$$

so that $A_1(x)$ is now transformed into x^k. The equation $y = x^k$ is transformed into $y = kx$, by replacing y by e^y and x by e^x, while $y = kx$ is of the form Type I*b*.

The Transforming Series as a Limit Series.

8. We shall now suppose that an $A_1(x)$ and a $G_1(x)$ are given and known, while the existence of an $F_1(x)$ is known but not the form of the series, where

$$A_1(x) \equiv F\{G_1[F_{-1}(x)]\}.$$

We shall also suppose that the general iterate $G_n(x)$ is known, and we wish to determine F so that

$$A_n(x) \equiv F\{G_n[F_{-1}(x)]\}.$$

We shall now describe a method which in some important cases serves to define an $F(x)$. The method is not always applicable but is of considerable interest in the cases in which it can be used.*

For any given positive integer p, let us consider the four series $G_{-p}[A_p(x)]$, $G_p[A_{-p}(x)]$, $A_{-p}[G_p(x)]$, $A_p[G_{-p}(x)]$, of which the latter two are the inverses respectively of the former two. A given term in any one of these series will be a function of p, which may or may not remain finite as p takes on successively larger and larger positive integer values. If it should happen that in one of these series the coefficient of each term approaches a definite finite limit for $p = +\alpha$, then a limit series, $L(x)$ will be defined. The question is one of convergence, not of the series as a series in x, but of the coefficients of the series as functions of p. If $L(x)$ exists, its inverse also exists, so that we may without loss of generality suppose L to be the limit series of one of the first two mentioned above. From the definition of $L(x)$ it then follows that

$$G[L(x)] \equiv L[A(x)],$$

so that $L(x)$ is a series of the form $F_{-1}(x)$ which was to be found. In determining $L(x)$ we have not made use of the actual form of $G_n(x)$, for n other than an integer, so that when different methods of interpolation are used to give different forms for $G_n(x)$, correspondingly different forms will be obtained for $A_n(x)$, n other than an integer. If $G_{-p}[A_p(x)]$ and $G_p[A_{-p}(x)]$ both have limit series L, and these are distinct, then for every integer value of n, we shall have

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* Cf. G. Koenigs, Ann. Ec. Norm. (3), 1 (1884), Suppl., p. 19, and (3) 2 (1885), p. 385.

$$G_n[L(x)] \equiv L[A_n(x)]$$

true, for both choices of $L(x)$, but for fractional, irrational or complex values of n, it may well happen that the same determination of $G_n(x)$ gives for the two L's different determinations of $A_n(x)$, which corresponds to the arbitrary feature of $G_n(x)$. If $C(x)$ be any series commutative with $G(x)$, then $C[L(x)]$ may be used equally well, in place of $L(x)$. In particular $C(x)$ may be an integral or any fractional iterate of $G(x)$. But this corresponds merely to having considered $G_{-p+q}[A_p(x)]$ or $G_{p-q}[A_{-p}(x)]$, respectively, in place of the given series, where q is a fixed number. Similarly we might have taken $L[C(x)]$ in place of $L(x)$, where $C(x)$ is this time commutative with $A(x)$. These two cases are not, of course, independent, since, as we have already remarked, $G_n[L(x)] \equiv L[A_n(x)]$. The one-parameter family of series $G_q[L(x)]$, q arbitrary, which exists if $L(x)$ does, constitutes in many cases the totality of series $F_{-1}(x)$.

Investigation of the Convergence of the Series Employed.

9. We now come to the question of convergence. In Type Ia, whenever $A_1(x)$ is a convergent series, the convergence of $E(x)$ follows, and conversely, as may be proved in several ways.* Type Ib, although not usually treated, presents no difficulties, since $A_1(x)$ and $A_{-1}(x)$ yield the same $E(x)$, and since when $|a_{11}| > 1$, then $|a_{-1,1}| < 1$, $a_{-1,1}$ being merely $1/a_{11}$. Thus, if $A_1(x)$ is convergent and $|a_{11}| > 1$, then $E(x)$ is convergent, because of the convergence of $A_{-1}(x)$. Type Ic is of a different sort. For $A_1(x)$ convergent and of Type Ic, $E(x)$ may be either convergent or divergent. If $A_1(x)$ is convergent certain conditions must be satisfied in order that $E(x)$ be convergent. These are not formal conditions upon the first p coefficients, $p = 1, 2, \cdots$, since the problem is one having significance only for convergent series $A_1(x)$, and convergence is not dependent upon the values of any finite number of coefficients. If $E(x)$ be convergent, $A(x)$ must be. Necessary and sufficient conditions for the convergence of $E(x)$, $A_1(x)$ being given as convergent, may be given in different forms. We shall state a condition which theoretically is capable of explicit application in any case. The cases of Type Ic, for which $E(x)$ is convergent, we shall call of Type Ic$'$, those for which $E(x)$ is divergent, being of Type Ic$''$, provided that $A_1(x)$ is convergent, otherwise of Type Ic$'''$. We shall now state

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* For example, Leau treats the equation in this instance both by the method of dominant functions, and by that of successive substitutions, loc. cit.

A necessary and sufficient condition for Type Ic'. It must be possible to find a circle about the origin in the complex plane such that if any point P other than the origin be taken within this circle, the totality of the positive and negative iterates of P, constitute a set of points lying everywhere densely upon a simple closed curve of finite length, and with continuously turning tangent, encircling the origin.

This includes, in particular, the conditions of interior and exterior "stability," about the origin. The condition of interior stability, frequently called merely "stability," is the following: For any sufficiently small circle C_1 with center at the origin, there must be a smaller concentric circle C_2, such that the positive and negative iterates of every point within C_2 is within C_1.* The condition for exterior stability insures that every iterate of a point in the neighborhood of the origin but exterior to a small circle enclosing the origin remains exterior to the same or a smaller fixed circle about the origin.

To prove that the condition stated above is sufficient, we may notice, that if it be satisfied for a given P, we shall have a curve C completely determined by the iterates of P, together with the conditions that C is continuous, and its tangent turns continuously. This curve C may be transformed into a circle in the plane of a complex variable t by a transformation of the form $t = E_{-1}(x)$, where $E_{-1}(x)$ is an analytic function, as follows from the theory of conformal mapping. The function E_{-1} will be completely determined when we require the origin in the x-plane to go into the origin in the t-plane, the circle in the t-plane being the unit circle and the positive real axis in the x-plane going into some sort of curve tangent to the positive real axis in the t-plane, at the origin. Any analytic transformation of the x-plane in the neighborhood of the origin, which carries the curve C into itself and the origin into itself, will correspond analytically to a transformation of the t-plane in the neighborhood of the origin, leaving the unit circle and the origin unaltered. Now $A_1(x)$ is such a transformation of the x-plane, and it corresponds therefore to a rotation in the t-plane, since a rotation is the only conformal transformation of the unit circle into itself which leaves the center invariant.† But formally it corresponds to a rotation $a_{11} \cdot t$, $|a_{11}| = 1$, where a_{11} is the initial coefficient of the series $A_1(x)$. Thus the formal series $E(x)$ is actually the inverse of the series corresponding to the analytic function $E_{-1}(x)$ just determined. Hence the above condition is sufficient; that it is necessary is seen immediately from the fact that if E is analytic then the successive iterates of a point P in the x-plane correspond to the successive images of a point in the t-plane, as the t-plane is rotated through an angle incommensurable with 2π.

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* Cf. Levi Civita, Ann. di. Mat. (3), 5 (1901), p. 240, and Cigala, Ann. di. Mat. (3), 11, (1905), p. 67.

† Cf. Osgood, Funktionentheorie, vol. I, 1st ed., p. 595–.

In the case of Type IIa, we may transform x into itself by the use of divergent as well as of convergent series. No simple or direct proof of the convergence of the series transforming Type IIb into

$$y = \frac{x}{1 - 2\tilde{a}_{12}x},$$

seems to have been made. The convergence follows, however, when $A_1(x)$ is itself, convergent, from the detailed investigation of a function-theoretic character made by Leau.* Since the series used in Type IIc are not formally equivalent to

$$y = \frac{x}{1 - 2\tilde{a}_{12}x},$$

no convergent power series can be found which transforms the former into the latter. The existence of functions in general analytic, but possessing essential singularities at the origin is proved by Leau for the cases in which the given series is convergent.

The existence of divergent series of the Type IIIa has already been shown. Whenever the series is convergent, the h series, u_i, $i = 0, 1, \cdots, h - 1$, are convergent, since they are linear combinations of the first h iterates of $A_1(x)$, so that the corresponding $E(x)$ must be necessarily convergent, while for $A_1(x)$, divergent, $E(x)$ is divergent. In the cases of Types IIIa and IIIb, for which, in general, no iterate containing an arbitrary parameter n, can exist of Case III, it is not surprising that no transformation is known which reduces either of these types to the form cx, or $x + a_0$, and certainly no conformal transformation of this sort can exist, so that there is no question of convergence.

A discussion of the convergence of the transforming function E, which reduces Type IVb, to the form $y = x^k$ has been made by Grèvy, *loc. cit.*

A Geometric Interpretation of Iteration.

10. We have already mentioned an infinite matrix whose first principal minor of order m, we have denoted by $\mathbf{A}_1^{(m)}$, which may be considered as serving to define the series $A_1(x)$. We have mentioned the problem of the reduction of this infinite matrix to a normal form by successively reducing to normal form the matrices $\mathbf{A}_1^{(m)}$, $m = 1, 2, \cdots$. In place of the collineation symbolized by $y = \mathbf{A}_1^{(m)}x$, and its reduction to a normal form, we may consider the entirely equivalent problem of the simultaneous linear transformation of the pair of bilinear forms, or of the λ-matrix, corresponding to the original matrix. In terms of the original series, $A_1(x)$, this means that we may consider the analytic transformation of the pair of

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* Leau, loc. cit.

expressions $y = x$, and $y = A_1(x)$, or of the linear function of λ, expressed by $y = A_1(x) - \lambda x$. Since any analytic curve in the real plane may be transformed analytically in the neighborhood of a non-singular point of the curve into $y = x$, the invariants under analytic transformations of a pair of curves at a point of intersection, are the invariants of the pair $y = x$, and $y = A_1(x)$, where $A_1(x)$ is the transform of the second curve. This excludes, of course, certain exceptional cases, as when a derivative of $A_1(x)$ becomes infinite, which, however, may be readily treated as limiting cases. The group of transformations, $y' = \varphi(y)$, $x' = \varphi(x)$, has significance chiefly in the " function-plane," and by change of variables, in the "inversion-plane." The study of the conformal invariants of a pair of real curves at a real point of intersection, is essentially the study of Cases II and III, discussed above, while the general conformal group without regard to reality involves also Cases I and IV.* Integral iteration corresponds to the multiplication of the angle between $y = A_1(x)$ and $y = x$, while division of the angle is secured by fractional iteration.

PART TWO. THE ITERATION OF A REAL FUNCTION.

The Domain of Definition of A_n.

11. We shall consider the iteration of a real function defined for the points within an interval a, b, $a < b$, where, in particular, a may be negatively infinite, or b positively infinite, or both. For many problems in analysis, it is immaterial whether we consider a real analytic function as defined also for imaginary points, or not, so far as the phenomena for real points alone are concerned. Iteration is not, however, one of these problems, since it frequently happens that the iterate of an imaginary point is itself a real point.

Let $A_1(x)$ be a finite, real, one-valued function of x, defined over a range r_1 in the interval a, b. We shall usually suppose that the range r_1' of values taken on by $x' = A_1(x)$ lies within the interval a, b, since otherwise, we might have extended our original interval a, b, so as to include r_1', without altering r_1 or any other data.

The function $A_2(x) \equiv A_1[A_1(x)]$, will be defined only for the numbers of the subset r_2 common to r_1 and r_1'. If we denote by r_2', the values assumed by $A_1(x)$ as x varies over r_2, then $A_3(x)$ is defined only for the subset r_3, common to r_2 and r_2', and so forth. It may happen that after iterating a finite number of times, we may eventually find an n, so large that r_n is a null-set, and no expression $A_n(x)$ can be defined. This case, although common, presents but little of interest. If r_1 be an open set, it may happen that for every positive integer n, r_n contains some number, while no number is common to all r_n's, owing to the fact that the limits approached lie outside of r_1. A third case arises when the limit set, r_∞, consists of a finite number of points, $e_1, e_2, \cdots, e_k$, so that these are the only points for which $A_n(x)$ can be defined for every positive integer n. Now $A_1(e_i)$ must be equal to one of the e's, say e_j, for if $A_n(e_i)$ is defined for all positive integral values of n, so must also $A_{n-1}[A_1(e_i)]$, since these are identical by definition, or in other words $A_1(e_i)$, is an e by definition. Since $A_1(x)$ is one-valued, each e_i must go into a unique e_j either itself or another. Of the k, e's, $e_1, e_2, \cdots, e_k$, it may happen that several distinct e's are carried into a single one by the operation $x' = A_1(x)$. There must be a non-vanishing subset of these k, e's into each one of which subset, an e of the original set is carried by $x' = A_1(x)$. By repeated selections of such subsets, we obtain finally a non-vanishing set of k', e's, which we may call $e_1, e_2, \cdots, e_{k'}$, such that each e of this set is carried into an e of this set and is obtained from an e of this set by any transformation $x' = A_n(x)$, n, a positive integer. Since k' is finite, and each e_i is carried into but a single e_j, each e_j must be obtained from but a single e_i of the set of k', e's. We have then a permutation group upon k' elements, and the transformation $x' = A_{k'!}(x)$ must leave each of this set unaltered. Other cases arise in which r_∞ coincides with r_1. It may possess no fixed points, a finite number, or an infinite number, and every point of r_∞ may be fixed.

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* Cf. Kasner, "Conformal Geometry," loc. cit., and Pfeiffer, notice in Bull. Am. Math. Soc., 21 (1915).

We may distinguish in all cases, as we have already done in one case, a subset of r_∞, which we shall call r_∞', where not only is $A_n(x)$ defined for every point of r_∞', for n an arbitrary positive integer, but $A_{-n}(x)$ is also defined for all positive integral values of n. For the subset of r_∞ obtained by removing r_∞', we have points which are carried into other points by A_n, but which are not obtainable from other points by an A_n, where n is a sufficiently large positive integer, in each case. The subset denotable by $r_\infty - r_\infty'$, may be further subclassified. Regarded as a set of points it is completely arbitrary in character. The set r_∞', may be divided into subsets, $r_{\infty 1}'$, $r_{\infty 2}'$, $\cdots$, together with certain residual sets $r_{\infty}{}'_{,\text{ell}}$, $r_\infty{}'_{\text{par}}$ and $r_\infty{}'_{,\text{hyp}}$. The set $r_\infty{}'_p$, for p a positive integer, is defined as a set which is carried pointwise into itself by $A_p(x)$, but by no $A_n(x)$, $n < p$. The set $r_\infty{}'_{,\text{ell}}$ is carried into itself circularly, but no point is carried into itself exactly by any A_n, n, a positive integer. This corresponds to the case of a rotation through an irrational angle. The set $r_\infty{}'_{,\text{hyp}}$, is a set carried into itself by $A_n(x)$, n a positive integer, but not pointwise, there being for any particular case two limit points, after the manner of a hyperbolic linear fractional transformation. Similarly any particular set $r_\infty{}'_{,\text{par}}$ has one limit point. The set $r_{\infty 1}'$ is absolutely arbitrary in form. The set $r_{\infty 1}'$ is restricted to the extent that it must consist of a set of pairs of points, without repetition, or as it may be regarded, as a pair of sets of points, the points in one set being in one to one reciprocal correspondence with those of the other. Similarly the set $r_{\infty n}$ must consist of sets of n points without repetition, or n sets in one to one reciprocal correspondence with one another. The sets, $r_{\infty}{}'_{,\text{ell}}$, may be regarded as constituting independent systems, as also in the cases of $r_{\infty}{}'_{,\text{par}}$ and $r_{\infty}{}'_{,\text{ell}}$.

Some General Theorems on A_n.

12. We shall state a few of the numerous theorems which are valid for real one-valued functions defined over a range r in an interval $a, b,\ a < b$, extending, in particular cases, to infinity. The method of proof is fairly obvious in every case, and as the proof itself is never difficult, it will be omitted. The index n, of iteration will be restricted to positive integers. We shall use the notation c, d_1 for an interval, $c \leq x \leq d_1$ included within the given interval.

If $A_1(x)$ is periodic, $A_n(x)$ is periodic in so far as it is defined.

Every point of intersection of $y = A_1(x)$ with $y = x$, is also a point of intersection of $y = A_n(x)$ with $y = x$.

If $c_1 \leq A_1(x) \leq d_1$, whenever $c_2 \leq x \leq d_2$, for $c_2 \leq c_1 < d_1 \leq d_2$, then the same is true for $A_n(x)$.

If $0 \leq s_1 < s_2$, and $A_1(e) = e$, and if also for x in the interval c, d, where $c \leq e \leq d$, $A_1(x)$ lies between $e + s_1(x - e)$ and $e + s_2(x - e)$, then for x in the same interval, $A_n(x)$ will lie between $e + s_1^n(x - e)$ and $e + s_2^n(x - e)$

If e be a value for which $A_1(e) = e$, then every point of intersection of $y = A_1(x)$ and $y = e$ is also a point of intersection of $y = A_n(x)$ and $y = e$.

Hence we obtain also the following theorem,

If $c \leq A_1(x) \leq d$, for $a \leq x \leq b$, and if $y \equiv x$ for $c \leq x \leq d$, then $A_n(x) \equiv A_1(x)$.

If for every $x_1 < x_2$, $A_1(x) < A_1(x)_2$, then also $A_n(x_1) < A_n(x_2)$, in so far as $A_n(x)$ is defined.

If the absolute minimum of $A_1(x)$ is unique and occurs at c, and $A_1(c) > c$ and also the absolute maximum of $A_1(x)$ is unique and occurs at d, and $A_1(d) < d$, and if $A_1(x)$ in so far as it is defined is an increasing function for $c \leq x \leq d$, then whenever $A_n(c)$ is defined it will be the unique absolute minimum value of $A_n(x)$, and $A_n(c) > A_{n-i}(c)$, $i = 1, 2, \cdots, n$, in so far as these are defined. A corresponding situation holds for $A_n(d)$, and $A_n(x)$ will be, in so far as it is defined, an increasing function of x for $c \leq x \leq d$.

If for every $x_1 < x_2$, $A_1(x_1) > A_1(x_2)$, then also $A_{2n-1}(x_1) > A_{2n-1}(x_2)$, while $A_{2n}(x_1) < A_{2n}(x_2)$ in so far as $A_{2n-1}(x)$ and $A_{2n}(x)$ are defined.

If $A_1(a)$ is either undefined or is equal to c_1 and for $a \leqq x_1 < x_2 \leqq c_2$, $c_1 \leqq A_1(x_1) < A_1(x_2) \leqq b$, and if for $c_2 \leqq x_1 < x_2 \leqq b$, $a \leqq A_1(x_1) < A_1(x_2) \leqq c_1$, while $A_1(b)$ is either undefined or equal to or less than c_1, then there exists a pair of number $c_1^{(n)}$ and $c_2^{(n)}$, for which an analogous situation holds for $A_n(x)$.

If in the preceding theorem we put "$\geqq$" in place of "$\leqq$", and $2n - 1$ for n, the theorem continues to hold, but if we put "$\geqq$" in the hypothesis and $2n$ in the conclusion, we must read "$\leqq$" in the conclusion.

Other theorems might be mentioned of a nature analogous to those just cited. We shall now however give a few of a different type, in which the continuity of $A_1(x)$ is supposed, as well as the existence and in some cases the continuity of the first derivative of $A_1(x)$ with respect to x. The proofs are obvious and will be omitted. They involve chiefly the use of the identity,

$$A_n'(x) = A_1'[A_{n-1}(x)] \cdot A_1'[A_{n-2}(x)] \cdots A_1'[A_1(x)] \cdot A_1'(x),$$

where $A_n'(x)$ denotes $\dfrac{\partial A_n(x)}{\partial x}$.

If the total number of maxima of $A_1(x)$ be finite and equal to m, while $A_1'(x)$ does not remain zero over any interval, the least possible number of maxima of $A_n(x)$ will be $m - 1$, and the greatest possible number will be $2^{n-1}m^n$. Examples of the two cases are most readily grasped by means of figures. Fig. 1 represents a broken line joining in order the points whose coordinates are, $[0, 1/(m - 1)]$, $[1/(m - 1), 0]$, $[2/(m - 1), 1/(m - 1)]$, $[3/(m - 1), 0]$, $\cdots$, $[2 - 1/(m - 1), 0]$, $[2, 1/(m - 1)]$. Odd iterates of

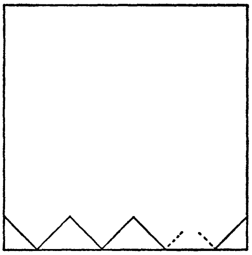

FIG. 1.

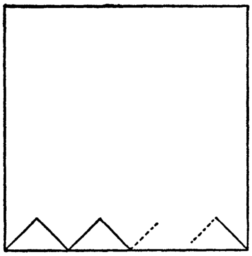

FIG. 2.

this curve coincide with the curve itself while even iterates are of the form illustrated in Fig. 2, where we have a broken line joining in order the points whose coordinates are $[0, 0]$, $[1/(m-1), 1/(m-1)]$, $[2/(m-1), 0]$, $[3/(m-1), 1/(m-1)]$, $\cdots$, $[2-1/(m-1), 1/(m-1)]$, $[2, 0]$. An example of the second case is given in Fig. 3, by the broken line joining in order, the points, $[0, 0]$, $[1/m, 2]$, $[2/m, 0]$, $[3/m, 2]$, $[4/m, 0]$, $\cdots$, $[2-1/m, 2]$, $[2, 0]$. The above theorem holds also if " minimum " be read for " maximum."

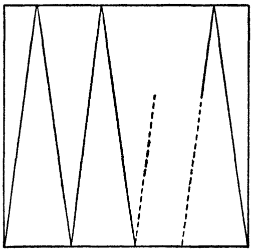

FIG. 3.

If for every x, $|A_1'(x)| < s$, where s is a constant, than $|A_n'(x)| < s^n$, throughout the interval. Hence if $y = A_1(x)$, be a curve of limited variation, so also is $y = A_n(x)$. The length of arc of $y = A_n(x)$ between $x = c$ and $x = d$, $c < d$, will be less than $(d-c)s^n$.

If $0 < s_1 < A_1'(x) < s_2$, then $0 < s_1^n < A_n'(s) < s_2^n$.

If $|A_1'(x)| \leqq 1$, throughout the interval, then the number of maxima of $A_n(x)$ cannot differ by more than one from the number of maxima of $A_1(x)$, provided that $A_1'(x)$ does not remain equal to zero over an interval. The same theorem holds true if " minima," be read for " maxima."

If throughout the interval, $A_1(x) \leq a + (x-a)^k/(b-a)^{k-1}$, then $A_n(x) \leq a + (x-a)^{k^n}/(b-a)^{k^n-1}$. The same theorem holds if "$\leq$," be replaced by "$\geq$."

One-Valued Iteration, A_n, for n Varying Continuously.

13. Let us examine those cases in which $A_1(x)$ is one-valued, and $A_n(x)$ exists as n varies continuously over real values. We shall have given a range r over which $A_1(x)$ is defined. In r, we shall distinguish two classes of points, which we shall call *singular points*, and *ordinary points*. An ordinary point will be defined as a point of r, for which there is no next point of r whether to the right or to the left of it, the other points are the singular points. A point which is not of r, but which is a limit point of points of r, and for which on both sides, there is no next point of r, will be called a *gap point* of r. We shall suppose that except in the neighborhood of $x = a$, and $x = b$, $A_n(x)$ exists for all values of n for which $|n|$ is suffi-

ciently small. Since for n irrational, $A_n(x)$, is defined only by means of interpolation, we shall be forced to regard $A_n(x)$ for any given x in r, as varying continuously over r, when n varies continuously over real values. This does not mean that $A_n(x)$ for a given x varies continuously in the interval a, b, as n varies continuously, but simply that it varies continuously with reference to the inner structure of r. Since continuous variation from a singular point is impossible, we must have $A_n(x) = x$, for every singular point. It must be borne in mind that a point is singular or not, purely in terms of the internal structure of r, and the end point of an interval c, d of points of r need not be a singular point. Since by continuous variation in r, a gap point may be approached but not passed, we see that the limit points approached by $A_n(x)$, for x given, and n increasing by integers toward infinity include in general gap points. If we take three numbers, $x_1 < x_2 < x_3$, then for $|n|$ small, we must have $A_n(x_1) < A_n(x_2) < A_n(x_3)$, provided, at least, that $a < x_1$ and $x_3 < b$, since a continuous group of transformations of the form here considered cannot include a transformation by which sense is altered, except possibly in the neighborhoods of a and b, for $|n|$ small. We may make three possible conventions with regard to the end points a and b, which will be in accord with the conditions imposed upon interior points.

First. We may require $A_n(x)$ to be defined at a and b only when n is positive for one of these points and negative for the other.

Second. We may require $A_n(x)$ to be defined at both a and b for $|n|$ small, and for n both positive and negative.

Third. We may regard the interval a, b, as treated cyclically, so that a and b are not distinguished, and a point to the right of a is treated as to the right of any point to the left of b, only cyclic order being essential. In this case, we may, if desired, require, in particular, that r coincides with the whole interval a, b, and $A_n(x)$ is defined for every point of the interval, whenever $|n|$ be small, whether n be positive or negative.

In the first case, a and b will be essentially different from any other points of r, if they, or either one, be in r. The range of definition for $A_n(x)$, will in general depend upon n, and various small difficulties arise. It is however a common and important case. In the second case, a and b, if in r, are merely singular points, as already defined, and are not to be distinguished from the other singular points. In the third case, a and b may be regarded as a single point in no way distinguishable from any ordinary point or r, either in terms of the structure of r itself, or in terms of the behavior of $A_n(x)$, as a function of n.

We may describe the curve $y = A_1(x)$, where $A_1(x)$ satisfies the conditions of this section as being *with respect to* r, a series of monotonically increasing curvilinear segments, and discrete points. The discrete points are all on the line $y = x$, and have for their abscissas, the singular points of r, measured on the x-axis. The curvilinear segments, if they have end points, have these on the line $y = x$. If we require that the range of definition interior to a, b, with the possible exception of the neighborhoods of a and b themselves, shall not vary with n, then we have also that curvilinear segments, where there is no end point, approach, i. e., at a gap point, the line $y = x$, and have a point of $y = x$ for limit point. When referred to the interval a, b, without the intervention of r, a more complicated description of $y = A_1(x)$, is, of course, necessary.

The Transforming Function $B(x)$.

14. In defining $A_n(x)$ for n, a fraction, we may make use of a transforming function, $B(x)$, such that if $y = A_n(x)$

$$\begin{cases} x = B(t), \\ y = B(n + t), \end{cases}$$

so that $A_n(x) \equiv B[n + B_{-1}(x)]$. If $y = A_n(x)$ is to be a one-valued function of x, for positive values of n, then $B(t)$ must be monotonic if it be continuous, that is, if $A_n(x)$ be a continuous function of n for a given x. For suppose if possible that $B(t)$ is one-valued and continuous but not monotonic. Then let $B(\alpha + t_1) = B(t_1)$, whence since $A_\alpha[B(t_1)] \equiv B(\alpha + t_1)$, we have $A_\alpha[B(t_1)] = B(t_1)$ and therefore $A_{m\alpha}[B(t_1)] = B(t_1)$, where m is any positive integer, or $B[m\alpha + t_1] = B(t_1)$, $m = 1, 2, \cdots$. $B(t)$, being supposed non-monotonic, will have either a maximum or a minimum, if not many of each. Let us suppose there is a maximum $\beta = B(\gamma)$. A line $x = \beta - \epsilon$, $\epsilon > 0$, will cut the curve, $x = B(t)$, at two points in the neighborhood of (γ, β), which we may call $t = \gamma - \delta_1$, and $t = \gamma + \delta_2$, $\delta_1 > 0$, $\delta_2 > 0$. Making use of our previous remark, we conclude that

$$B[m(\delta_2 + \delta_1) + \gamma - \delta_1] = B(\gamma - \delta_1), \qquad m = 1, 2, \cdots.$$

Thus the line $x = \beta - \epsilon$ contains a set of points of the curve $x = B(t)$ beginning with $t = \gamma - \delta_1$, and succeeding each other at a constant interval $\delta_1 + \delta_2$. As ϵ is taken smaller and smaller, we obtain new lines and each contains points of the curve $x = B(t)$ distributed in the same fashion at constant intervals from each other, the constant interval, however, decreasing with ϵ. From the continuity of B we would conclude that in the limiting case of the line, $x = \beta$ every point of this line to the right of $t = \gamma$, ought to be a point of the curve $x = B(t)$. But if $x = B(t)$ is to be one-valued, such a situation is impossible. We therefore conclude, that for $x = B(t)$, continuous and one-valued, this curve is also monotonic. The proof for the case of a minimum but no maximum follows in the same fashion and is therefore supposed completed.

The range of the variable t, has, of course, nothing to do with the range r or the interval a, b. If $B(t)$ is monotonic and one-valued, its most significant characteristics will be its vertical and horizontal asymptotes if one or both of these exist. If $u = B(t)$ has a horizontal asymptote $u = e$, then $x = e$, is a fixed point for $y = A_n(x)$. If $u = B(t)$ has a vertical asymptote, then $y = A_n(x)$ has for every n, a fixed common horizontal asymptote, and the interval a, b must be infinite, while if the curve $u = B(t)$ be defined to the left of a vertical asymptote, then $y = A_n(x)$ has a vertical asymptote. Further theorems of a like sort are readily obtained, for instance: If $u = B(t)$ approaches minus infinity, as t approaches e, from the left, then $y = A_n(x)$, is an increasing function with a vertical asymptote $x = e - 1$, and lying to the right of the asymptotes.

A transforming function $B(t)$ must be itself monotonic and is adapted for the study of only monotonic functions $y = A_1(x)$. If in the neighborhood of a point $x = y = e$, $y = A_1(x)$ is monotonic and increasing, we may obtain for this neighborhood a transforming function $x = B(t)$. The question arises as to whether if we continue $A_1(x)$ beyond the neighborhood in which it is monotonic, there will not be a corresponding extension of $B(t)$, so that the $B(t)$ so extended shall still serve as the transforming function of $y = A_1(x)$ when this latter is extended. We shall find that no extension of $B(t)$ can be made to serve as a transforming function for $A_1(x)$ when extended beyond the region in which $A_1(x)$ is monotonic. If, however, $A_1(x)$ is analytic, we may determine $B(t)$ analytically, and each will be continued analytically beyond the region in which it is monotone. What then is the relation between these continuations? For $y = A_1(x)$, one-valued, but not monotonic, the extension beyond the monotonic interval containing a point $x = y = e$, leads us to an extension of $B(t)$ into an *auxiliary* part, such that x and y may both vary on the *principal* part $B(t)$ with the relation $x = B(t), y = B(1 + t)$, but x alone varies on the auxiliary part $x = \tilde{B}(t)$, while y continues to satisfy $y = B(1 + t)$. The situation may readily be grasped upon inspecting the case in which $A(x) \equiv \sin x$. The curve $u = B(t)$ is indicated, Fig. 4, by a heavy curve, while the

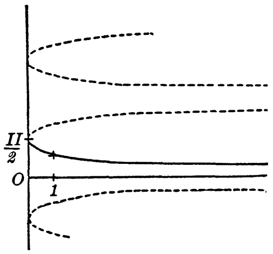

FIG. 4.

auxiliary part $u = \tilde{B}(t)$, here consists of an infinite sequence of branches a few of which are here indicated by dotted curves.

No inverse $A_{-1}(x)$, in a strict sense, exists when $A_1(x)$ is non-monotonic. For example, the equation $y = \text{arc sin}\,[\sin x]$, is satisfied not only by $y = x$, but also by $y = x + 2m\pi$, and $y = (2m + 1)\pi - x$, where m is an arbitrary integer. For these reasons the study of the complete behavior in the complex plane of a transforming function $B(t)$, for a given $A(x)$, does not present the interest that might be at first expected.

Multiple-Valued Functions. Weights.

15. We shall now consider the case of multiple-valued real functions. Let $y = A_1(x)$, $y = F_1(x)$, $y = G_1(x)$, $\cdots$, be a set of single-valued functions, which taken in their totality constitute a multiple-valued function $y = \varphi_1(x)$. The expression $y = \varphi_2(x)$ may be regarded as an abridged way of writing the set of one-valued functions

$$\begin{array}{llll} y = A_2(x), & y = A_1[F_1(x)], & y = A_1[G_1(x)], & \cdots, \\ y = F_1[A_1(x)], & y = F_2(x), & y = F_1[G_1(x)], & \cdots, \\ y = G_1[A_1(x)], & y = G_1[F_1(x)], & y = G_2(x), & \cdots. \end{array}$$

In most cases of multiple-valued functions φ, the expression, $y = \varphi_n(x)$, can have no meaning for n other than a positive integer. It sometimes happens that two or more of the original one-valued components of $y = \varphi_1(x)$, are commutative, *e. g.* perhaps $F_1[A_1(x)] \equiv A_1[F_1(x)]$. In this case, $y = A_1[F_1(x)]$, occurs twice in the expression $y = \varphi_2(x)$. This requires us to consider the curves as *weighted*, each curve having a positive integer *weight*, corresponding to the number of times that it is to be counted.

One of the most important cases of a multiple-valued real function is the very special case in which $\varphi(x)$ consists of two parts, one of which is simply $y = x$, and the other of which is a real monotonic, increasing, single-valued function, $y = A_1(x)$. We shall obtain in this case,

$$\begin{aligned} \varphi_0(x) &\equiv x \equiv A_0(x), \\ \varphi_1(x) &\equiv \{A_0(x), A_1(x)\}, \\ \varphi_2(x) &\equiv \{A_0(x), 2A_1(x), A_2(x)\}, \\ \varphi_3(x) &\equiv \{A_0(x), 3A_1(x), 3A_2(x), A_3(x)\}, \\ &\cdots\cdots\cdots\cdots\cdots \\ \varphi_n(x) &= \{A_0(x), \tbinom{n}{1}A_1(x), \tbinom{n}{2}A_2(x), \cdots, \tbinom{n}{i}A_i(x), \cdots\}, \end{aligned}$$

for n a positive integer. By extending the notion of weight so as to include negative, fractional, and irrational values of the weight, we may define

$\varphi_n(x)$ for all real values of n by the relation

$$\varphi_n(x) \equiv \{A_0(x),\ \binom{n}{1}A_1(x),\ \binom{n}{2}A_2(x),\ \cdots,\ \binom{n}{i}A_n(x),\ \cdots\},$$

the expressions $\binom{n}{1}$ indicating as usual the coefficients in a binomial expansion. It is, furthermore, only by some such extension that the expression $\varphi_{-1}(x)$ can be determined or even defined.

Matrix Interpretation.

16. The iteration of a real function derives a great deal of added significance when regarded as a matrix multiplication. The matrices $\boldsymbol{A}_n^{(m)}$ which we have hitherto considered have contained but a discrete set of elements. We shall now consider matrices of a more general type, in which the elements that appear in the matrix may form continuous systems. One such type of matrix is suggested by the $\boldsymbol{A}_1^{(m)}$ already considered. The elements of the $(i + 1)$st row of $\boldsymbol{A}_1^{(m)}$, are the coefficients of $A_1(x^i)$, where the coefficient of x^j appears in the $(j + 1)$st column, i and j being integers. If now we allow i and j to take on all real positive values, we shall obtain an analogous matrix, in which zeros occur everywhere except on the main diagonal and lines parallel to it starting with the columns at integral units distances from the initial element, and in the first row.

We shall now introduce certain functions $M_n(x, y)$ of two variables, x and y, which functions we shall call *matrices*. The matrix $M_1(x, y)$ associated with a single valued function $y = A_1(x)$, will be the function which vanishes for points not on the locus $y = A_1(x)$, but is unity on the locus. More generally, for a multiple-valued and arbitrarily weighted function $y = \varphi_1(x)$, we shall mean by $M_1(x, y)$ the function whose value at any point (x, y) is equal to the weight of the point, where points not on any component of $y = \varphi_1(x)$ are considered as of weight zero. The matrix product of $M_1(x, y)$ by itself we shall call $M_2(x, y)$ and in general the matrix product of $M_{n-1}(x, y)$ by $M_1(x, y)$ we shall denote by $M_n(x, y)$. The value of $M_2(x, y)$ for any point (x, y) is defined as the algebraic sum of the values of the algebraic product of $M_1(x, t)$ by $M_1(t, y)$, as t varies over the interval a, b, in accordance with the usual notion of matrix product extended in an obvious manner.

If we write $y = \varphi_1(t)$, and $t = \varphi_1(x)$, and give x and y arbitrarily selected values, y_1 and x_1 respectively, the roots of $y_1 = \varphi_1(t)$, $t = (t_1, t_2, t_3, \cdots)$ are in general all distinct from the values of $t = \varphi_1(x_1)$, $t = (t_1', t_2', t_3', \cdots)$. Let however the common values, if any, be denoted by $t_1'', t_2'', t_3'', \cdots$. The weight of the point (t_i'', y_1) we may denote by α_i, $i = 1, 2, \cdots$, and the weight of the point (x_1, t_i''), by β_i, $i = 1, 2, \cdots$. To obtain $y = \varphi_2(x)$, we must eliminate t from $y = \varphi_1(t)$ and $t = \varphi_1(x)$. In particular the weight of the point (x_1, y_1) will be the sum of the weights given by the points, t_1'', t_2'', t_3'', $\cdots$, or in other words it will be $\Sigma_i\alpha_i\beta_i$. Hence the process of determining φ_2 is much the same as that of finding $M_2(x, y)$; in fact the matrix $M_2(x, y)$ is exactly the matrix M corresponding to $y = \varphi_2(x)$, and, in general, the matrix of $\varphi_n(x)$ is exactly $M_n(x, y)$.

We may now find an entirely new interpretation to one of the formulas already given. Let us denote by $a_{11}z$ a matrix which vanishes except for $x = y$, for which equality, it assumes arbitrary values along this diagonal. By $V_1(z)$ we shall denote a matrix which vanishes along the main diagonal, *i. e.* for $x = y$. By $A_1(z)$ we shall denote the matrix which is the sum of $a_{11}z$ and $V_1(z)$. We shall regard z as a symbolic variable upon which we are operating with the matrices a_{11}, V_1, A_1, etc. The matrix product of the operation $a_{11}z$ by itself we shall denote by a_{11}^2z and the inverse of $a_{11}z$ by $(1/a_{11})z$, etc. We shall then have the formula (5) as holding also with this new interpretation, namely,

$$A_n(x) \equiv a_{11}^n\left[V_0(z) + \binom{n}{1}\frac{1}{a_{11}}V_1(z) + \binom{n}{2}\frac{1}{a_{11}^2}V_2(z) + \cdots + \binom{n}{i}\frac{1}{a_{11}^i}V_i(z) + \cdots\right],$$

where

$$V_i(z) \equiv A_i(z) - \binom{i}{1}a_{11}A_{i-1}(z) + \binom{i}{2}a_{11}^2A_{i-2}(z) + \cdots \pm a_{11}^iA_0(z)$$

and where in the present instance $V_i(z)$ is the ith iterate of $V_1(z)$. It is particularly important as a formula, when $V_1(z)$ represents a curve which is one-valued and monotonic. The weight of $V_1(x)$ may vary from point to point. When a_{11} is equal to unity along the main diagonal and the weight of V_1 is either one or zero, V_1 being the matrix of a one-valued monotonic function, we reobtain the formula given in § 15 for $\varphi_n(x)$.

We have interpreted the iteration of a real function as finding the power of a matrix. It is also possible to interpret finding the power of an ordinary matrix as a case of iteration. For concreteness let us consider the example of the following normal matrix:

$$A_1 \equiv \begin{vmatrix} r_1 & s_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_1 & s_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 & s_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_3 \end{vmatrix}.$$

Here we have for a_{11}

$$a_{11} \equiv \begin{vmatrix} r_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_3 \end{vmatrix},$$

while V_1 is

$$V_1 \equiv \begin{vmatrix} 0 & s_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Here V_2 contains but a single element different from zero, and V_3, $V_4 \cdots$, vanish identically. From the formula, we obtain

$$A_n \equiv \begin{vmatrix} r_1^n & nr_1^{n-1}s_{11} & \frac{n(n-1)}{2} r_1^{n-2}s_{11}s_{12} & 0 & 0 & 0 & 0 \\ 0 & r_1^n & nr_1^{n-1}s_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2^n & nr_2^{n-1}s_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2^n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2^n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_3^n \end{vmatrix},$$

which may be readily verified.*

PART THREE. SOME PROPERTIES OF CERTAIN RATIONAL FUNCTIONS UNDER ITERATION.

A Special Class of Rational Functions.

17. One of the simplest and most interesting classes of functions $y = A_1(x)$ to study in the complex plane, is that of the rational functions.

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* Cf. Van Vleck, "One Parameter Projective Groups . . . ," Trans. Am. Math. Soc., 13 (1912), p. 353–386.

Certain rational functions with real coefficients have the property that the iterates of the real function are identical with the iterates considered as real functions. In these cases the positive and negative iterates of a real point are themselves real. It is a class of these functions that we shall now consider.

Let us be given a rational function $A_1(x)$ of degree r in numerator and denominator. For one value of x, $y = A_1(x)$, will have but one value, while for y assigned arbitrarily x must have r values. If these are to be real whenever y is real, which we must require in order that real points are carried into real points by A_{-1}, then the r zeros of $A_1(x)$ must be real and distinct; since if $y = 0$ yields two coincident roots, then $y = c$, where c is properly selected with respect to sign, and different from zero, will have two imaginary roots. Furthermore and for the same reason $y = A_1(x)$ can have no horizontal tangent at a finite point, so that between any two zeros of $y = A_1(x)$, there must be a pole of this function, and this continues to be true when we regard the range cyclically. Thus the poles and zeros alternate. The curve $y = A_1(x)$ must then either increase monotonically between every pair of vertical asymptotes, or else decrease monotonically between every such pair. In either case the expression $\dfrac{dA_1(x)}{dx}$ must have the same sign at all of the zeros of $A_1(x)$. The straight line $y = x$ meets the curve $y = A(x)$ in $r + 1$ points real or imaginary. If the curve decrease monotonically between asymptotes, that is, if $[dA_1(x)]/dx$ is negative at the zeros of $A_1(x)$, then these $r + 1$ points of intersection must all be real and distinct, while for the curve increasing between asymptotes, $r - 1$ at least must be real and distinct, these lying between successive vertical asymptotes, while the two remaining are either conjugate imaginary, real and coincident, or real and distinct.

The sum of the residues of $1/[A_1(x) - x]$, must be zero. We shall suppose that the points of intersection $e_1, e_2, \cdots, e_{r+1}$, of $y = x$ with $y = A_1(x)$ are all finite. We may then obtain a relation among the slopes of the curve at the points e, by use of the residues of $1/[A_1(x) - x]$. The curve $y = A_1(x)$, will be supposed to have r finite poles, and hence it must have a real horizontal asymptote, say, $y = \alpha$. If $1/[A_1(x) - x]$ be expanded in descending powers of x, its first coefficients will be given by using the expression of the form

$$\frac{1}{A_1(x) - x} = -\frac{1}{x - \dfrac{P(x)}{Q(x)}} = \frac{1}{x}\left[\frac{-1}{1 - \dfrac{\dfrac{\alpha}{x} + \dfrac{b_1}{x^2} + \dfrac{b_2}{x^3} + \cdots}{1 + \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \cdots}}\right] = -\frac{1}{x} - \frac{\alpha}{x^2} + \cdots$$

which determines the residue at infinity of $1/[A_1(x) - x]$ as being equal to unity. If for $dA_1(x)/dx$ we write $A_1'(x)$, the residue at e_i will be $1/[A_1'(e_i)-1]$, hence we have,

$$\sum_{i=1}^{r+1} \frac{1}{1 - A_1'(e_i)} = 1.$$

If $A_1(x)$ decreases between the vertical asymptotes, then $A_1'(x)$ is certainly negative in particular for the $r + 1$ points e_i, $i = 1, 2, \cdots, r + 1$. If it is increasing between the asymptotes, then $A_1'(x)$ is real and greater than unity at each of $r - 1$ points e, one lying between each pair of vertical asymptotes. If the two remaining e's are real and distinct then $A_1'(x)$ will be greater than unity at one of them, and positive but less than unity at the other.

In case $A_1'(x)$ is negative at the points e, there can be no question of a real one-valued function $y = A_n(x)$ which is a continuous function of n, and which leaves one of the e's invariant. For if such an $A_n(x)$ existed its slope at any e, i. e., $A_n'(x)$ would be equal to $[A_1'(x)]^n$ at the same point, but this cannot be real and vary continuously with n and still be negative for $n = 1$. We shall examine now in greater detail the case when a function $y = A_n(x)$ might exist and vary continuously with n, i. e., the case in which $A_1'(x)$ is positive at the real points e. We shall confine ourselves also to the case in which not only $r - 1$, e's are real and distinct but in which the two remaining e's are real and all $r + 1$ are distinct. The cases in which the two remaining e's are conjugate imaginary or coincident may be treated in an analogous manner and present no new features of special interest.

The Iteration of Certain Rational Functions.

18. If we be given a rational function, $y = A_1(x)$, we may, as we have stated before, replace x by a linear fractional transform of x, and y by the same linear fractional transform of y, and the transform of A_1 obtained in this manner has for its iterates the transforms of the iterates of the original A_1. By taking a real linear fractional operation we may arrange matters in such a way that $r - 1$, e's, $e_1, e_2, \cdots, e_{r-1}$, occur each between a pair of vertical asymptotes, while e_r and e_{r+1} occur to the right of the right-most asymptote.

We shall suppose therefore that we are given two polynomials $P(x)$ and $Q(x)$ of degree r, and we shall write $A_1(x) \equiv P(x)/Q(x)$. We shall suppose P and Q to be real, and the r zeros of $P(x)$, $z_1, z_2, \cdots, z_r$ to be real, and the r poles of $A_1(x)$, i. e., the r zeros of $Q(x)$, $p_1, p_2, \cdots, p_r$, to be real while $- \infty < p_1 < z_1 < p_2 < z_2, \cdots, < p_r < z_r < + \infty$. We shall also suppose that $A_1'(z) > 0$ for each point z. There will be $r - 1$ points e_i, $i = 1, 2,$

$\cdots, r-1$, for which $A_1(e_i) = e_i$, and such that $p_i < e_i < p_{i+1}$, while $A_1'(e_i) > 0$. The two remaining roots of $A_1(x) = x$, e_r, and e_{r+1}, we may denote also by f_1 and f_2 respectively. We shall suppose these to be real and distinct, and satisfying the inequality $p_r < f_1 < f_2$. Then $A_1'(f_1) > 1$, and $0 < A_1'(f_2) < 1$.

For n a positive integer greater than one, the function $y = A_n(x)$ has many features analogous to those of $y = A_1(x)$. It is a rational, real function of degree r^n, with r^n real zeros and r^n real poles, and the poles and zeros alternate. The roots of $A_n(x) = x$ are $r^n + 1$ in number of which $r^n - 1$ are real and occur one each between every pair of poles, while the last two are real and to the right of the right-most pole, and coincide indeed with f_1 and f_2. The left-most pole of $A_n(x)$ is to the right of the left-most pole of $A_{n-1}(x)$ but lies also in the interval p_1, p_2. The right-most pole of $A_n(x)$ lies to the right of the right-most pole of $A_{n-1}(x)$ but also in the interval $p_r f_1$. The function $y = A_n(x)$ increases monotonically between every pair of poles, while $A_n'(f_1) > 1$ and $0 < A_n'(f_2) < 1$. If α_1 denotes the asymptotic value of $y = A_1(x)$, then the asymptotic value, α_n, of $A_n(x)$, is such that $\alpha_n < \alpha_1$.

For n positive but nonintegral, the functions $A_n(x)$ which we can determine in the neighborhood of f_1 and of f_2 coincide, in the sense that the series valid at f_1 when analytically extended must be valid at f_2, and must coincide with the series obtained at f_2, for no singularity can occur between these two points. It might be at first supposed that by further continuation we should obtain a function with essential singularities, perhaps, on the real axis, but none the less one-valued in the complex plane. This cannot however be the case if n be allowed to vary continuously, as we shall proceed to show by the use of the Riemann surface for $A_1(x)$. That $y = A_n(x)$ is not one-valued when $n = -1$, and r (the degree) is greater than one, is obvious.

Let us mark in the complex plane of x, the branch points of $y = A_{-1}(x)$, and join these by cuts, to obtain a region R_0, which is simply connected and includes all points of the complex plane not themselves on the cuts. Since f_1 is not itself a branch point we shall suppose that no cut actually passes through f_1, and likewise for f_2. Now the interior of a small circle about f_2 is carried by the transformation $y = A_{-1}(x)$ into the interior of a curve approximately circular and concentric with the given circle, since f_2 is carried into itself, and $A_{-1}'(f_2)$ is real and greater then unity. So that any small region, X, inclosing f_2 is carried into a region Y entirely including X within its interior. Now $A_{-1}(x)$ is defined as one-valued, either finite or infinite, for every x in R_0, so that as X is successively extended Y suffers a like extension and always so as to include X, as long as X is within R_0.

As X comes to coincide with R_0, Y is, of course, no longer a non-overlapping region but coincides with the Riemann surface for $y = A_{-1}(x)$, except for certain cuts, a region which we shall call R_{-1}. We shall define R_{-2} as the multiple-sheeted region that Y becomes when X is taken as R_{-1}, and in general for $X \equiv R_{-n}$ we shall have $Y \equiv R_{-(n+1)}$. On the other hand, for $Y \equiv R_0$, X becomes a region included within R_0 and only partially covering the complex plane. This we shall call R_1. We shall define analogously R_2, R_3, $\cdots$. Now we have a series of regions,

$$\cdots, R_3, R_2, R_1, R_0, R_{-1}, R_{-2}, R_{-3}, \cdots$$

each included in the following, and building up successively an infinitely sheeted surface, which we shall call R.

By the discussion made in the first part of this paper, we may obtain the coefficients of a power series $E(x)$ which represents an analytic function, such that $E[A_{-1}(x)] = (1/a_{11})E(x)$. If we write $t = E(x)$, then the transformation $y = A_{-1}(x)$ corresponds in the t-plane to the transformation $t' = (1/a_{11})t$ where $t' = E[A_{-1}(x)]$. Thus the mapping of R_n upon R_{n-1} has for its transform in the t-plane, a mere magnification leaving the origin invariant. The straight lines through the origin in the t-plane are carried into themselves by the magnification, and are the path curves of the transformation in the t-plane for $t' = (1/a_{11}{}^n)t$. The corresponding path curves in the x-plane, are the path curves for $A_{-n}(x)$, as n varies over real values, and are curves starting at f_2 and passing, in their totality, through every point of R. For $t = E(x)$, as t varies over the t-plane, x varies over R, so that R is the Riemann surface for $t = E(x)$. Now let us take a number x. There will be one point in each sheet of the infinitely-sheeted surface R, having x for its coordinate. Through each of these points there will be a path curve terminating at f_2. To determine $A_n(x)$, for a given real, positive n, we must follow along a path curve from x toward f_2 a distance corresponding to n in the scale that is determined along the path curve by its very nature. The infinite number of points which we shall obtain by starting with the same x but in different sheets, can only be superposed for all choices of x and n when the path curves in the different sheets of R are all superposed. This is not, however, the case unless the rational function be linear fractional, since otherwise $y = A_{-1}(x)$ would carry a single x into but a single y. Hence $A_n(x)$ is not a one-valued function of x as n varies over real positive values.

The function $x = E_{-1}(t)$ is a one-valued meromorphic function of t with infinity as an essential singularity, while $E(x)$ itself is an infinitely many-valued function with R for its Riemann surface.

The same sort of discussion will show that if $A(x)$ be an analytic function defined with its inverse, all over the plane and $A_1(x)$ is one-valued, then while $A_n(x)$ cannot be one-valued for general values of n, still the inverse E_{-1} of the transforming function E is in general one-valued and meromorphic over the complex plane, provided that there is one point of f used in defining E for which $A_1(f) = f$, and $|A_1'(f)| = A_1'(f) < 1$. We shall not investigate less stringent conditions at this time.

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